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A DISTORTION THEOREM FOR BIHOLOMORPHIC MAPPINGS IN $\mathbb{C}^2$

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ABSTRACT. Let J_f be the Jacobian of a normalized biholomorphic mapping f from the unit ball B^2 into $\mathbb{C}^2$. An expression for the $\log\det J_f$ is determined by considering the series expansion for the renormalized mappings F obtained from f under the group of holomorphic automorphisms of B^2. This expression is used to determine a bound for $|\det J_f|$ and $|\arg\det J_f|$ for f in a compact family X of normalized biholomorphic mappings from B^2 into $\mathbb{C}^2$ in terms of a bound $C(X)$ of a certain combination of second-order coefficients. Estimates are found for $C(X)$ for the specific family X of normalized convex mappings from B^2 into $\mathbb{C}^2$.

1. INTRODUCTION

Distortion theorems for families of univalent functions have been studied at least since 1907 when Köbe discovered his classical "Verzerrungsatz": the distortion theorem for the class of univalent functions defined on the unit disc in the complex plane $\mathbb{C}$. Köbe's theorem gives explicit upper and lower bounds on $|f'(z)|$ in terms of $|z|$ for any function f that is one-to-one and analytic on the unit disc and normalized by $f(0) = 0$ and $f'(0) = 1$. The term distortion arises from the geometric interpretation of $|f'(z)|$ as the infinitesimal magnification factor of arc length and the interpretation of the square of $|f'(z)|$ as the infinitesimal magnification factor of area.

In P. Montel's book on univalent function theory, Henri Cartan wrote an appendix titled "Sur la possibilité d'éntendre aux fonctions de plusieurs variables complexes la théorie des fonctions univalentes" in which he called for a number of generalizations of properties of univalent functions in one variable to biholomorphic mappings in several variables. He specifically cited the special classes of starlike and convex mappings as appropriate topics for generalization. (We have followed his suggestions concerning starlike mappings elsewhere [1].) Cartan indicated particular interest in the properties of the determinant of the complex Jacobian of univalent mappings (the square of the magnitude of the determinant of the complex Jacobian is the infinitesimal magnification factor of volume in C^n). He stated a "théorème présumé" that the magnitude of the determinant of the Jacobian of a normalized biholomorphic mapping would have a finite upper and a positive lower bound depending only on $|z| = r < 1$ and indicated the real interest and merit in determining these bounds. That his conjecture did not hold was probably known for some time. A discussion of

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Cartan's note and a simple class of counterexamples to his conjecture appears in [4]. We include the following counterexample. For any positive integer k, let $f(z) = (f_1(z), f_2(z))$ with

$$\begin{cases} f_1(z) = z_1, \\ f_2(z) = z_2(1 - z_1)^{-k} = z_2 + kz_2z_1 + \cdots. \end{cases} \tag{1}$$

Then f is a normalized biholomorphic function on the unit ball B^2 in $\mathbb{C}^2$, that is, $f(0) = 0$, and the Jacobian of f at the origin is the identity matrix. The Jacobian of f is given by

$$J_f = \begin{pmatrix} 1 & 0 \\ \dfrac{kz_2}{(1 - z_1)^{k+1}} & \dfrac{1}{(1 - z_1)^k} \end{pmatrix}.$$

Thus $|\det J_f| = |1 - z_1|^{-k}$ giving

$$\max_{|z|\le r} |\det J_f| = (1 - r)^{-k} \to \infty \quad \text{as } k \to \infty$$

and

$$\min_{|z|\le r} |\det J_f| = (1 + r)^{-k} \to 0 \quad \text{as } k \to \infty.$$

We note that the counterexample also applies for z in the polydisc $\{(z_1, z_2): |z_1| < 1, |z_2| < 1\}$ as well as the unit ball. We also note from (1) that there is no bound on the magnitude of second-order coefficients of normalized univalent functions in the polydisc or in B^2. The lack of a bound on these coefficients shows that the usual proofs of one variable of Köbe's distortion theorem do not extend to $\mathbb{C}^n$ for $n \geq 2$. In one dimension, the usual proofs use the bound of the magnitude of the second coefficient for normalized univalent functions and the invariance of the full class of univalent functions under Möbius transformations.

That Cartan's conjectured distortion theorem does not hold in the class of normalized biholomorphic mappings in $\mathbb{C}^n$, $n \geq 2$, suggests the problem of finding a subset for which the conjecture does hold. We are thus led to consider the class of normalized convex mappings which Cartan had already suggested for study. For this class the coefficients of second-order terms are of bounded magnitude. With considerable alterations, we will extend the one variable proof to obtain a distortion theorem in $\mathbb{C}^2$.

As mentioned, the invariance of the full class of univalent functions under Möbius transformations has been a powerful tool in obtaining theorems and other results concerning the subclass of normalized univalent functions. Also in several variables, invariance has played an important role. In [7, p. 278], W. Rudin introduced the term "M-invariant space" to mean a space of holomorphic mappings on B^2 in $\mathbb{C}^2$, which are invariant under holomorphic automorphisms of the ball B^2 in $\mathbb{C}^2$. That is, if a mapping f on the ball belongs to the space, then f composed with ψ belongs to the space for every ψ in $\mathrm{Aut}(B^2)$. Our main theorem will be for compact subsets of M-invariant spaces of biholomorphic mappings on the ball B^2 in $\mathbb{C}^2$. The M-invariance of convexity is used to obtain estimates in the subclass of normalized convex mappings of B^2.

Characterizations of convex mappings in $\mathbb{C}^n$ have been obtained by Suffridge in [8] and later by Kikuchi in [6]. Suffridge characterizes convex mappings both on the unit ball B^n as well as the polydisc, $E = \{z \in \mathbb{C}^n : |z_1| < 1, |z_2| < 1, \dots, |z_n| < 1\}$. He also proved the rather surprising result that a holomorphic mapping from the polydisc E, a reducible domain, maps onto a convex domain in $\mathbb{C}^n$ if and only if there exist univalent functions f_k $(1 \le k \le n)$ from the unit disc in the plane onto convex domains in the plane such that $f(z) = T(f_1(z_2), f_2(z_2), \dots, f_n(z_n))$ where T is a nonsingular linear transformation. Hence the standard distortion results follow in a straightforward manner in this case. However, in the unit ball B^2, an irreducible domain, this type of reduction does not occur and thus our interest in the ball for this class. Indeed, E. Cartan has shown that every bounded, transitive domain in $\mathbb{C}^2$ is biholomorphically equivalent to either a polydisc or the ball; thus, once we obtain our estimates for the case of the ball, the problem is essentially solved for all bounded transitive domains in $\mathbb{C}^2$.

2. Main results

We now indicate the technique for estimation of the magnitude of the logarithm of the determinant of the Jacobian of a mapping f at a. The rate of change of this magnitude with respect to the components of the vector a is bound by certain combinations of the second-order coefficient of a particular new mapping F. This new mapping is obtained by composing f with a holomorphic automorphism of the ball and then renormalizing. We are then able to give explicit bounds for the logarithm of the determinant of the Jacobian in terms of bounds on this combination of the second-order coefficients, provided these bounds exist. In §4 we determine an estimate for these bounds for the class K of normalized convex mappings on B^2. In the final section we make a conjecture as to sharp bounds and give a supporting example. We concentrate our considerations to mappings from B^2 into $\mathbb{C}^2$ noting that extensions to B^n can be made in principle albeit with even more formable computations.

For an explicit statement of our main result we let points in $\mathbb{C}^2$ be denoted by row vectors $z = (z_1, z_2)$ for z_1, z_2 in $\mathbb{C}$. A prime on a vector will indicate the transpose. The origin $(0, 0)$ is sometimes denoted by 0. Mappings f in $\mathbb{C}^2$ are given as row vectors $f = (f_1, f_2)$ where each function f_i is a function in $\mathbb{C}^2$. The Jacobian of a holomorphic mapping f is the matrix with its ijth entry $\partial f_i/\partial z_j$ and is denoted by J or J_f. A holomorphic mapping f is normalized means that $f(0) = 0$ and $J_f(0) = I$ the identity matrix. A mapping f can be normalized by application of a complex affine transformation: $[f(z) - f(0)]J'^{-1}$ where J is evaluated at the origin. Let f be defined by $f(z) = (f_1(z), f_2(z))$ with

$$
\begin{aligned}
f_1(z) &= z_1 + d^{(1)}_{2,0} z_1^2 + d^{(1)}_{1,1} z_1 z_2 + d^{(1)}_{0,2} z_2^2 + \cdots , \\
f_2(z) &= z_2 + d^{(2)}_{2,0} z_1^2 + d^{(2)}_{1,1} z_1 z_2 + d^{(2)}_{0,2} z_2^2 + \cdots .
\end{aligned}
\tag{2}
$$

Applying the result in [5] we let $\phi_a(w)$ be the Möbius transformation of B^2 onto B^2 taking $a = (a_1, a_2)$ into the origin where ϕ_a will be explicitly given in (7); then for $z = (z_1, z_2)$ fixed in B^2 and $(a_1, a_2) = (\rho z_1, \rho z_2)$, $0 < \rho \le 1$, we define G by $G(w) = f[\phi_a(w)]$ and renormalize so that the resulting normalized mapping F has the expansion

$$F(w) = (w_1, w_2) + (w_1^2, w_1w_2, w_2^2)D + \cdots$$

where

$$D = \begin{pmatrix} d^{(1)}_{2,0}(\rho), & d^{(1)}_{1,1}(\rho), & d^{(1)}_{0,2}(\rho) \\ d^{(2)}_{2,0}(\rho), & d^{(2)}_{1,1}(\rho), & d^{(2)}_{0,2}(\rho) \end{pmatrix}'. \tag{3}$$

Then letting J denote the Jacobian of f at z we have

(4)

$$\log\det J = \log\frac{1}{(1-z\bar{z}')^{3/2}} - 2\int_0^1 \frac{z_1(d^{(1)}_{2,0} + (1/2)d^{(2)}_{1,1}) + z_2(d^{(2)}_{0,2} + z_2(d^{(2)}_{0,2} + (1/2)d^{(1)}_{1,1})}{1-\rho^2 z\bar{z}'}\, d\rho$$

where we have dropped the indicated dependency of the d's on ρ. Then we use a unitary transformation and (4) to obtain the main result:

Theorem. *Consider an M-invariant space S of biholomorphic mappings on the unit ball B^2 into $\mathbb{C}^2$. Let X be the mappings obtained from mappings of S by normalizing by a complex affine transformation. Suppose that X is compact. With the notation of equations* (2), *define $C(X)$ by*

$$C(X) = \sup\{|d^{(1)}_{2,0} + (1/2)d^{(2)}_{1,1}|: f \in X\}. \tag{5}$$

For any mapping f in X, let J denote its Jacobian at z. Then

$$|\log[(1-z\bar{z}')]^{3/2}\det J| \le C(X)\log\left(\frac{1+|z|}{1-|z|}\right).$$

As a consequence we have

$$\frac{(1-|z|)^{C(X)-3/2}}{(1+|z|)^{C(X)+3/2}} < |\det J| < \frac{(1+|z|)^{C(X)-3/2}}{(1-|z|)^{C(X)+3/2}} \tag{6}$$

and

$$|\arg\det J| \le C(X)\log\left(\frac{1+|z|}{1-|z|}\right)$$

where the branch of the argument is chosen to be zero at the origin.

In §4 we determine that for $X = K$, the class of normalized convex functions in B^2,

$$C(K) < 1.761.$$

In §5 we conjecture that $C(K) = 3/2$, give a supporting example, and suggest the corresponding sharp bounds in (6) for $\mathbb{C}^n$, for each $n \ge 2$.

3. Biholomorphic mappings under the group of biholomorphic automorphisms

In this section we show how the expression (4) is obtained.

In [5] Sheng Gong and Z. Yan proved a very general result involving the covariant derivative of the Bergman metric which gives a closed form expansion for the composition of a holomorphic mapping with an automorphism of a bounded domain in $\mathbb{C}^n$. For our case the mapping is from B^2 to $\mathbb{C}^2$. We carry out the initial calculations for a (not necessarily normalized) mapping $g: B^2 \to \mathbb{C}^2$. Let $z = (z_1, z_2)$, $w = (w_1, w_2)$, and $a = (a_1, a_2) \in B^2$. (Since equation (4) is obviously true for $z = (0, 0)$, we can assume that $z \neq (0, 0)$ and $a \neq (0, 0)$ in the following calculations.) Then a Möbius transformation $\phi_a(w): B^2 \to B^2$ taking a into the origin is given by

$$z = \phi_a(w) = \frac{a - P_a z - sQ_a z}{1 - z\overline{a}'}$$

where P_a is the orthogonal projection of $\mathbb{C}^2$ onto the subspace $[a]$ generated by a, $Q_a = I - P_a$, I the identity matrix, so that Q_a is the projection onto the orthogonal complement of $[a]$ and $s = \sqrt{1 - a\overline{a}'}$ with $\overline{a}'$ denoting the conjugate transpose of the row vector a. More explicitly we can write

$$\phi_a(w) = \frac{a - w}{1 - w\overline{a}'}A \tag{7}$$

where A is the matrix defined by

$$A = \frac{\overline{a}'a + s(a\overline{a}'I - \overline{a}'a)}{a\overline{a}'}. \tag{8}$$

This gives

$$\begin{aligned} z_1 &= \frac{a_1 a\overline{a}' - a_1 w\overline{a}' - s(a\overline{a}'w_1 - w\overline{a}'a_1)}{a\overline{a}'(1 - w\overline{a}')}, \\ z_2 &= \frac{a_2 a\overline{a}' - a_2 w\overline{a}' - s(a\overline{a}'w_2 - w\overline{a}'a_2)}{a\overline{a}'(1 - w\overline{a}')}. \end{aligned} \tag{9}$$

From the mapping g which is not necessary normalized, we define a composed $G: B^2 \to \mathbb{C}^2$ by

$$G(w) = g(z) = g[\phi_a(w)] = g\left[\frac{a - w}{1 - w\overline{a}'}A\right].$$

We now compute the initial power series expansion of $G(w)$.

$$\begin{cases} \dfrac{\partial G}{\partial w_1} = \dfrac{\partial g}{\partial z_1}\dfrac{\partial z_1}{\partial w_1} + \dfrac{\partial g}{\partial z_2}\dfrac{\partial z_2}{\partial w_1}, \\ \dfrac{\partial G}{\partial w_2} = \dfrac{\partial g}{\partial z_1}\dfrac{\partial z_1}{\partial w_2} + \dfrac{\partial g}{\partial z_2}\dfrac{\partial z_2}{\partial w_2}. \end{cases} \tag{10}$$

By (9) we have

$$\begin{aligned} \frac{\partial z_1}{\partial w_1} &= \frac{s(-|a_1|^2 s - |a_2|^2 + a\overline{a}'w_2\overline{a}_2)}{a\overline{a}'(1 - w\overline{a}')^2}, \\ \frac{\partial z_2}{\partial w_1} &= \frac{s(\overline{a}_1 a_2 - \overline{a}_1 a_2 s - a\overline{a}'w_2\overline{a}_1)}{a\overline{a}'(1 - w\overline{a}')^2}, \\ \frac{\partial z_1}{\partial w_2} &= \frac{s(\overline{a}_2 a_1 - \overline{a}_2 a_1 s - a\overline{a}'w_1\overline{a}_2)}{a\overline{a}'(1 - w\overline{a}')^2}, \\ \frac{\partial z_2}{\partial w_2} &= \frac{s(-|a_2|^2 s - |a_1|^2 + a\overline{a}'w_1\overline{a}')}{a\overline{a}'(1 - w\overline{a}')^2}. \end{aligned} \tag{11}$$

Moreover,

(12)
$$\frac{\partial^2 G}{\partial w_1^2} = \frac{\partial^2 g}{\partial z_1^2}\left(\frac{\partial z_1}{\partial w_1}\right)^2 + 2\frac{\partial^2 g}{\partial z_1 \partial z_2}\frac{\partial z_1}{\partial w_1}\frac{\partial z_2}{\partial w_1} + \frac{\partial^2 g}{\partial z_2^2}\left(\frac{\partial z_2}{\partial w_1}\right)^2 + \frac{\partial g}{\partial z_1}\frac{\partial^2 z_1}{\partial w_1^2} + \frac{\partial g}{\partial z_2}\frac{\partial^2 z_2}{\partial w_1^2},$$
$$\frac{\partial^2 G}{\partial w_1 \partial w_2} = \frac{\partial^2 g}{\partial z_1^2}\frac{\partial z_1}{\partial w_2}\frac{\partial z_1}{\partial w_1} + \frac{\partial^2 g}{\partial z_1 \partial z_2}\left(\frac{\partial z_1}{\partial w_1}\frac{\partial z_2}{\partial w_2} + \frac{\partial z_1}{\partial w_2}\frac{\partial z_2}{\partial w_1}\right) + \frac{\partial^2 g}{\partial z_2^2}\frac{\partial z_2}{\partial w_2}\frac{\partial z_2}{\partial w_1} + \frac{\partial^2 z_1}{\partial w_1 \partial w_2}\frac{\partial g}{\partial z_1} + \frac{\partial^2 z_2}{\partial w_1 \partial w_2}\frac{\partial g}{\partial z_2},$$
$$\frac{\partial^2 G}{\partial w_2^2} = \frac{\partial^2 g}{\partial z_1^2}\left(\frac{\partial z_1}{\partial w_2}\right)^2 + 2\frac{\partial^2 z}{\partial z_1 \partial z_2}\frac{\partial z_1}{\partial w_2}\frac{\partial z_2}{\partial w_2} + \frac{\partial^2 g}{\partial z_2^2}\left(\frac{\partial z_2}{\partial w_2}\right)^2 + \frac{\partial g}{\partial z_1}\frac{\partial^2 z_1}{\partial w_2^2} + \frac{\partial g}{\partial z_2}\frac{\partial^2 z_2}{\partial w_2^2}.$$

From (11) we get

(13)
$$\frac{\partial^2 z_1}{\partial w_1^2} = \frac{2s\overline{a}_1(-|a_1|^2 s + w\overline{a}' a\overline{a}' - |a_2|^2 - a\overline{a}' w_2 \overline{a}_2)}{a\overline{a}'(1 - w\overline{a}')^3},$$
$$\frac{\partial^2 z_1}{\partial w_1 \partial w_2} = \frac{s\overline{a}_2(|a_1|^2 - |a_2|^2 + a\overline{a}' w_2 \overline{a}_2 - a\overline{a}' w_1 \overline{a}_1 - 2|a_1|^2 s)}{a\overline{a}'(1 - w\overline{a}')^3},$$
$$\frac{\partial^2 z_1}{\partial w_2^2} = \frac{2s\overline{a}_2^2(a_1 - a_1 s - a\overline{a}' w_1)}{a\overline{a}'(1 - w\overline{a}')^3},$$
$$\frac{\partial^2 z_2}{\partial w_1^2} = \frac{2s\overline{a}_1^2(a_2 - a_2 s - a\overline{a}' w_2)}{a\overline{a}'(1 - w\overline{a}')^3},$$
$$\frac{\partial^2 z_1}{\partial w_1 \partial w_2} = \frac{s\overline{a}_1(|a_2|^2 - |a_1|^2 + a\overline{a}'(w_1\overline{a}' - w_2\overline{a}_2) - 2|a_2|^2 s)}{a\overline{a}'(1 - w\overline{a}')^3},$$
$$\frac{\partial^2 z_2}{\partial w_2^2} = \frac{2s\overline{a}_2(-|a_2|^2 s + w\overline{a}' a\overline{a}' - |a_1|^2 - a\overline{a}' w_2 \overline{a}_2)}{a\overline{a}'(1 - w\overline{a}')^3}.$$

We expand $G(w)$ at $w = 0$,

(14)
$$G(w) = G(0) + \left.\frac{\partial G}{\partial w_1}\right|_{w=0} w_1 + \left.\frac{\partial G}{\partial w_2}\right|_{w=0} w_2 + \frac{1}{2}\left.\frac{\partial^2 G}{\partial w_1^2}\right|_{w=0} w_1^2 + \left.\frac{\partial^2 G}{\partial w_1 \partial w_2}\right|_{w=0} w_1 w_2 + \frac{1}{2}\left.\frac{\partial^2 G}{\partial w_2^2}\right|_{w=0} w_2^2 + \cdots.$$

When $w = 0$ then $z = a$ and we let

(15)
$$c = s/a\overline{a}', \qquad \alpha = s|a_1|^2 + |a_2|^2, \qquad \beta = -\overline{a}_1 a_2(1 - s), \qquad \gamma = s|a_2|^2 + |a_1|^2.$$

Then (11) and (13) imply

(16)
$$\left.\frac{\partial z_1}{\partial w_1}\right|_{w=0} = -c\alpha, \qquad \left.\frac{\partial z_1}{\partial w_2}\right|_{w=0} = -c\overline{\beta}, \qquad \left.\frac{\partial z_2}{\partial w_1}\right|_{w=0} = -c\beta, \qquad \left.\frac{\partial z_2}{\partial w_2}\right|_{w=0} = -c\gamma$$

and

$$(17)\quad \begin{cases} \left.\dfrac{\partial^2 z_1}{\partial w_1^2}\right|_{w=0} = -2c\overline{a}_1\alpha, & \left.\dfrac{\partial^2 z_1}{\partial w_1 \partial w_2}\right|_{w=0} = c(-\overline{a}_2\alpha - \overline{a}_1\overline{\beta}), \\ \left.\dfrac{\partial^2 z_1}{\partial w_2^2}\right|_{w=0} = -2c\overline{a}_2\overline{\beta}, & \left.\dfrac{\partial^2 z_2}{\partial w_1^2}\right|_{w=0} = -2c\overline{a}_1\beta, \\ \left.\dfrac{\partial^2 z_2}{\partial w_1 \partial w_2}\right|_{w=0} = c(-\overline{a}_1\gamma - \overline{a}_2\overline{\beta}), & \left.\dfrac{\partial^2 z_2}{\partial w_2^2}\right|_{w=0} = -2c\overline{a}_2\gamma. \end{cases}$$

By (10), (12), (15), and (16) and by letting

$$\frac{\partial g}{\partial a_1} = \left.\frac{\partial g(z)}{\partial z_1}\right|_{z=a}, \qquad \frac{\partial^2 g}{\partial a_1^2} = \left.\frac{\partial^2 g(z)}{\partial z_1^2}\right|_{z=a}, \quad \text{etc.},$$

we have

(18)

$$\left.\frac{\partial G}{\partial w_1}\right|_{w=0} = -c\left\{\alpha\frac{\partial g}{\partial a_1} + \beta\frac{\partial g}{\partial a_2}\right\}, \qquad \left.\frac{\partial G}{\partial w_2}\right|_{w=0} = -c\left\{\overline{\beta}\frac{\partial g}{\partial a_1} + \gamma\frac{\partial g}{\partial a_2}\right\},$$

$$\left.\frac{\partial^2 G}{\partial w_1^2}\right|_{w=0} = c^2\left[\alpha^2\frac{\partial^2 g}{\partial a_1^2} + 2\alpha\beta\frac{\partial^2 g}{\partial a_1\partial a_2} + \beta^2\frac{\partial^2 g}{\partial a_2^2}\right] - 2c\overline{a}_1\left[\alpha\frac{\partial g}{\partial a_1} + \beta\frac{\partial g}{\partial a_2}\right],$$

(19)

$$\left.\frac{\partial^2 G}{\partial w_1\partial w_2}\right|_{w=0} = c^2\left[\alpha\overline{\beta}\frac{\partial^2 g}{\partial a_1^2} + (\alpha\gamma + |\beta|^2)\frac{\partial^2 g}{\partial a_1\partial a_2} + \beta\gamma\frac{\partial^2 g}{\alpha a_2^2}\right] - c\left[(\overline{a}_1\overline{\beta} + \overline{a}_2\alpha)\frac{\partial g}{\partial a_1} + (\overline{a}_2\beta + \overline{a}_1\gamma)\frac{\partial g}{\partial a_2}\right],$$

$$\left.\frac{\partial^2 G}{\partial w_2^2}\right|_{w=0} = c^2\left[\gamma^2\frac{\partial^2 g}{\partial a_2^2} + 2\gamma\overline{\beta}\frac{\partial^2 g}{\partial a_1\partial a_2} + \overline{\beta}^2\frac{\partial^2 g}{\partial a_2^2}\right] - 2c\left[\gamma\frac{\partial g}{\partial a_2} + \overline{\beta}\frac{\partial g}{\partial a_1}\right].$$

Substitution of (18) and (19) into equation (14) gives

$$(20)\quad \begin{aligned} g(z) = G(w) = g(a) &+ c\left[-\alpha\frac{\partial g}{\partial a_1} - \beta\frac{\partial g}{\partial a_2}\right]w_1 + c\left[-\overline{\beta}\frac{\partial g}{\partial a_1} - \gamma\frac{\partial g}{\partial a_2}\right]w_2 \\ &+ \frac{1}{2}\left[c^2\left(\alpha^2\frac{\partial^2 g}{\partial a_1^2} + 2\alpha\beta\frac{\partial^2 g}{\partial a_1\partial a_2} + \beta^2\frac{\partial^2 g}{\partial a_2^2}\right)\right. \\ &\qquad \left. + 2c\overline{a}_1\left(-\alpha\frac{\partial g}{\partial a_1} - \beta\frac{\partial g}{\partial a_2}\right)\right]w_1^2 \\ &+ \left\{c^2\left[\alpha\overline{\beta}\frac{\partial^2 g}{\partial a_1^2} + (\alpha\gamma + |\beta|^2)\frac{\partial^2 g}{\partial a_1\partial a_2} + \beta\gamma\frac{\partial^2 g}{\partial a_2^2}\right]\right. \\ &\qquad \left. + c\left[(-\overline{a}_1\overline{\beta} - \overline{a}_2\alpha)\frac{\partial g}{\partial a_1} + (-\overline{a}_2\beta - \overline{a}_1\gamma)\frac{\partial g}{\partial a_2}\right]\right\}w_1w_2 \\ &+ \frac{1}{2}\left[c^2\left(\alpha^2\frac{\partial^2 g}{\partial a_2^2} + 2\overline{\beta}\gamma\frac{\partial^2 g}{\partial a_1\partial a_2} + \overline{\beta}^2\frac{\partial^2 g}{\partial a_1^2}\right)\right. \\ &\qquad \left. + 2c\overline{a}_2\left(-\gamma\frac{\partial g}{\partial a_2} - \overline{\beta}\frac{\partial g}{\partial a_1}\right)\right]w_2^2 + \cdots. \end{aligned}$$

Now we consider the special case in which $g = f$, our normalized mapping. We continue to use G with components G_1, G_2 for the composed mapping so that

$$G(w) = f\left(\frac{a-w}{1-w\overline{a}'}A\right) = (G_1(w), G_2(w)).$$

Then by (20) with c, α, β, γ still defined as in (15), we have

$$\begin{aligned} G(w) - G(0) &= (f_1(z) - f_1(a), f_2(z) - f_2(a)) \\ &= c(w_1, w_2)\begin{pmatrix} -\alpha\frac{\partial f_1}{\partial a_1} - \beta\frac{\partial f_1}{\partial a_2}, & -\alpha\frac{\partial f_2}{\partial a_1} - \beta\frac{\partial f_2}{\partial a_2} \\ -\overline{\beta}\frac{\partial f_2}{\partial a_1} - \gamma\frac{\partial f_1}{\partial a_2}, & -\overline{\beta}\frac{\partial f_1}{\partial a_1} - \gamma\frac{\partial f_2}{\partial a_2} \end{pmatrix} + \cdots \\ &= c(w_1, w_2)\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix} J' + \frac{1}{2}(w_1^2, w_1w_2, w_2^2)\left(\frac{\partial^2 G}{\partial w^2}\right)_{w=0} + \cdots \end{aligned} \tag{21}$$

where J is the Jacobian of $f(z)$ at $z = a$ and

$$\left(\frac{\partial^2 G}{\partial w^2}\right)_{w=0} = \begin{pmatrix} \frac{\partial^2 G_1}{\partial w_1^2}, & \frac{\partial^2 G_2}{\partial w_1^2} \\ 2\frac{\partial^2 G_1}{\partial w_1 \partial w_2}, & 2\frac{\partial^2 G_2}{\partial w_1 \partial w_2} \\ \frac{\partial^2 G_1}{\partial w_2^2}, & \frac{\partial^2 G_2}{\partial w_2^2} \end{pmatrix}_{w=0}. \tag{22}$$

By (19), the right-hand side of (22) equals

$$c^2 M\left(\frac{\partial^2 f}{\partial a^2}\right) - 2cNJ' \tag{23}$$

where

$$M = \begin{pmatrix} \alpha^2, & 2\alpha\beta, & \beta^2 \\ 2\alpha\overline{\beta}, & 2(\alpha\gamma + |\beta|^2), & 2\beta\gamma \\ \overline{\beta}^2, & 2\overline{\beta}\gamma, & \gamma^2 \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f_1}{\partial a_1^2}, & \frac{\partial^2 f_2}{\partial a_1^2} \\ \frac{\partial^2 f_1}{\partial a_1 \partial a_2}, & \frac{\partial^2 f_2}{\partial a_1 \partial a_2} \\ \frac{\partial^2 f_1}{\partial a_2^2}, & \frac{\partial^2 f_2}{\partial_2^2} \end{pmatrix}, \tag{24}$$

$$N = \begin{pmatrix} \overline{\alpha}_1\alpha, & \overline{\alpha}_1\beta \\ \overline{\alpha}_2 + \overline{\alpha}_1\overline{\beta}, & \overline{\alpha}_1\gamma + \overline{\alpha}_2\beta \\ \overline{\alpha}_2\overline{\beta}, & \overline{\alpha}_2\gamma \end{pmatrix}.$$

Hence we have

$$\begin{aligned} G(w) - G(0) &= c(w_1, w_2)\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix} J' \\ &+ \frac{c^2}{2}(w_1^2, w_1w_2, w_2^2)M\left(\frac{\partial^2 f}{\partial a^2}\right) - c(w_1^2, w_1w_2, w_2^2)NJ' + \cdots. \end{aligned} \tag{25}$$

Next we normalize this mapping. This step will complete the plan indicated after equation (2) of transforming the function f by a Möbius transformation and then renormalizing. Multiplying (25) on the right by $\frac{1}{c}J'^{-1}\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix}^{-1}$, we have the normalized mapping $F(w)$ given by

$$\begin{aligned} F(w) &= \frac{1}{c}(G(w) - G(0))J'^{-1}\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix}^{-1} \\ &= (w_1, w_2) + \frac{c}{2}(w_1^2, w_1w_2, w_2^2)M\left(\frac{\partial^2 f}{\partial a^2}\right)J'^{-1}\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix}^{-1} \\ &\quad - (w_1^2, w_1w_2, w_2^2)N\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix}^{-1} + \cdots \\ &= (w_1, w_2) + (w_1^2, w_1w_2, w_2^2)D + \cdots \end{aligned} \tag{26}$$

with

$$D = \begin{pmatrix} d_{2,0}^{(1)}, & d_{2,0}^{(2)} \\ d_{1,1}^{(1)}, & d_{1,1}^{(2)} \\ d_{0,2}^{(1)}, & d_{0,2}^{(2)} \end{pmatrix}$$

where each component of D depends on $a = (a_1, a_2)$.

We now will express $(\partial^2 f/\partial a^2)J'^{-1}$ in terms of the coefficients of the second-order terms and the matrices depending on the point a. From (26) we obtain

$$D = \frac{c}{2}M\left(\frac{\partial^2 f}{\partial a^2}\right)J'^{-1}\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix}^{-1} - N\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix}^{-1}. \tag{27}$$

Multiplication of (27) on the left by $M^{-1}2/c$ and on the right by $\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix}$ gives

$$\frac{2}{c}M^{-1}D\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix} = \left(\frac{\partial^2 f}{a^2}\right)J'^{-1} - \frac{2}{c}M^{-1}N$$

which we write as

$$\left(\frac{\partial^2 f}{\partial a^2}\right)J'^{-1} = \frac{2}{c}M^{-1}D\begin{pmatrix} -\alpha, & -\beta \\ -\overline{\beta}, & -\gamma \end{pmatrix} + \frac{2}{c}M^{-1}N. \tag{28}$$

A direct computation of the inverse of M shows that

$$M^{-1} = \frac{1}{(a\overline{a}')^4(1 - a\overline{a}')}\begin{pmatrix} \gamma^2, & -\beta\gamma, & \beta^2 \\ -\gamma\overline{\beta}, & -\frac{1}{2}(\alpha\gamma + |\beta|^2), & -\alpha\beta \\ \overline{\beta}^2, & -\alpha\overline{\beta}, & \alpha^2 \end{pmatrix}.$$

From (15), it follows that

$$\alpha\gamma - |\beta|^2 = s(a\overline{a}')^2. \tag{29}$$

Using (29), one obtains

$$M^{-1}N = \frac{1}{(a\overline{a}')^2\sqrt{1 - a\overline{a}'}}\begin{pmatrix} \overline{a}_2\beta - \overline{a}_1\gamma, & 0 \\ \frac{1}{2}(\overline{a}_1\overline{\beta} - \overline{a}_2\alpha), & \frac{1}{2}(\overline{a}_2\beta - \overline{a}_1\gamma) \\ 0, & \overline{a}_1\overline{\beta} - \overline{a}_2\alpha \end{pmatrix}. \tag{30}$$

Thus

(31)

$$\left(\frac{\partial^2 f}{\partial a^2}\right)J'^{-1} = \frac{1}{\det J}\begin{pmatrix} \frac{\partial^2 f_1}{\partial a_1^2}\frac{\partial f_2}{\partial a_2} - \frac{\partial^2 f_2}{\partial a_1^2}\frac{\partial f_1}{\partial a_2}, & -\frac{\partial^2 f_1}{\partial a_1^2}\frac{\partial f_2}{\partial a_1} + \frac{\partial^2 f_2}{\partial a_1^2}\frac{\partial f_1}{\partial a_1} \\ \frac{\partial^2 f_1}{\partial a_1\partial a_2}\frac{\partial f_2}{\partial a_2} - \frac{\partial^2 f_1}{\partial a_1\partial a_2}\frac{\partial f_1}{\partial a_2}, & -\frac{\partial^2 f_1}{\partial a_2\partial a_1}\frac{\partial f_2}{\partial a_1} + \frac{\partial^2 f_2}{\partial a_1\partial a_2}\frac{\partial f_1}{\partial a_1} \\ \frac{\partial^2 f_1}{\partial a_2^2}\frac{\partial f_2}{\partial a_2} - \frac{\partial^2 f_2}{\partial a_1^2}\frac{\partial f_1}{\partial a_2}, & -\frac{\partial^2 f_1}{\partial a_2^2}\frac{\partial f_2}{\partial a_1} + \frac{\partial^2 f_2}{\partial a_2^2}\frac{\partial f_1}{\partial a_1} \end{pmatrix}$$

and

$$M^{-1}D\begin{pmatrix} -\alpha, & -\beta \\ -\beta, & -\gamma \end{pmatrix} = \frac{1}{(a\overline{a}')^4(1-a\overline{a}')}\begin{pmatrix} l_{11}, & l_{12} \\ l_{21}, & l_{22} \\ l_{31}, & l_{32} \end{pmatrix} \tag{32}$$

where

$$\begin{aligned}
l_{11} &= -\alpha\gamma^2 d^{(1)}_{2,0} + \alpha\beta\gamma d^{(1)}_{1,1} - \alpha\beta^2 d^{(1)}_{0,2} - \overline{\beta}\gamma^2 d^{(2)}_{2,0} + |\beta^2|\gamma d^{(2)}_{1,1} - \beta|\beta^2| d^{(2)}_{0,2}, \\
l_{12} &= -\beta\gamma^2 d^{(1)}_{2,0} + \beta^2\gamma d^{(1)}_{1,1} - \beta^3 d^{(1)}_{0,2} - \gamma^3 d^{(2)}_{2,0} + \beta\gamma^2 d^{(2)}_{1,1} - \beta^2\gamma d^{(2)}_{0,2}, \\
l_{21} &= \alpha\overline{\beta}\gamma d^{(1)}_{2,0} - \frac{\alpha}{2}(\alpha\gamma + |\beta^2|) d^{(1)}_{1,1} + \alpha^2\beta d^{(1)}_{0,2} + \gamma\overline{\beta}^2 d^{(2)}_{2,0} \\
&\quad - \frac{\beta}{2}(\alpha\gamma + |\beta|^2) d^{(2)}_{1,1} + \alpha|\beta|^2 d^{(2)}_{0,2}, \\
l_{22} &= \gamma|\beta^2| d^{(1)}_{2,0} - \frac{\beta}{2}(\alpha\gamma + |\beta^2|) d^{(1)}_{1,1} + \alpha\beta^2 d^{(1)}_{0,2} + \gamma^2\overline{\beta} d^{(2)}_{2,0} \\
&\quad - \frac{\gamma}{2}(\alpha\gamma + |\beta|^2|) d^{(2)}_{1,1} + \alpha\beta\gamma d^{(2)}_{0,2}, \\
l_{31} &= -\alpha\overline{\beta}^2 d^{(1)}_{2,0} + \alpha^2\overline{\beta} d^{(1)}_{1,1} - \alpha^3 d^{(1)}_{0,2} - \overline{\beta}^3 d^{(2)}_{2,0} - \alpha\overline{\beta}^2 d^{(2)}_{1,1} - \alpha^2\overline{\beta} d^{(2)}_{0,2}, \\
l_{32} &= -\overline{\beta}|\beta^2| d^{(1)}_{2,0} + \alpha|\beta^2| d^{(1)}_{1,1} - \alpha^2\beta d^{(1)}_{0,2} - \overline{\beta}^2\gamma d^{(2)}_{2,0} + \alpha\overline{\beta}\gamma d^{(2)}_{1,1} - \alpha^2\gamma d^{(2)}_{0,2}.
\end{aligned}$$

We now observe that

$$\begin{aligned}
\frac{\partial}{\partial a_1}\log\det J &= \frac{1}{\det J}\frac{\partial}{\partial a_1}\left[\frac{\partial f_1}{\partial a_1}\frac{\partial f_2}{\partial a_2} - \frac{\partial f_2}{\partial a_1}\frac{\partial f_1}{\partial a_2}\right] \\
&= \frac{1}{\det J}\left[\frac{\partial^2 f_1}{\partial a_1^2}\frac{\partial f_2}{\partial a_2} - \frac{\partial^2 f_2}{\partial a_1^2}\frac{\partial f_1}{\partial a_2} - \frac{\partial^2 f_2}{\partial a_1 \partial a_2}\frac{\partial f_2}{\partial a_1} + \frac{\partial^2 f_2}{\partial a_1 \partial a_2}\frac{\partial f_1}{\partial a_1}\right].
\end{aligned}$$

The last expression equals the sum of two entries of the right side of (31), specifically the sum of the $(1, 1)$ entry and the $(2, 2)$ entry. Since (31) and (28) restate the same matrix, we can use the right side of (28) as expressed by (30), (32), and (33). Adding entries $(1, 1)$ and $(2, 2)$ we obtain

$$\begin{aligned}
\frac{\partial}{\partial a_1}\log\det J = &\frac{2}{(a\overline{a}')^3(1-a\overline{a}')^{3/2}} \\
&\times\Big(-\alpha\gamma^2 d^{(1)}_{2,0} + \alpha\beta\gamma d^{(1)}_{1,1} - \alpha\beta^2 d^{(1)}_{0,2} - \overline{\beta}\gamma^2 d^{(2)}_{2,0} + |\beta^2|\gamma d^{(2)}_{1,1} \\
&\qquad - \beta|\beta^2| d^{(2)}_{0,2} + \gamma|\beta|^2 d^{(1)}_{2,0} - \frac{\beta}{2}(\alpha\gamma + |\beta^2|) d^{(1)}_{1,1} \\
&\qquad + \alpha\beta^2 d^{(1)}_{0,2} + \gamma^2\overline{\beta} d^{(2)}_{2,0} - \frac{\gamma}{2}(\alpha\gamma + |\beta|^2) d^{(2)}_{1,1} + \alpha\beta\gamma d^{(2)}_{0,2}\Big) \\
&+ \frac{2}{a\overline{a}'(1-a\overline{a}')}\left[\overline{a}_1\gamma - \overline{a}_2\beta + \frac{1}{2}(\overline{a}_1\gamma - \overline{a}_2\beta)\right].
\end{aligned}$$

When the last expression is simplified with the use of (29), we find that

$$\frac{\partial}{\partial a_1}\log\det J = \frac{3\overline{a}_1}{1-a\overline{a}'} + \frac{2}{a\overline{a}'(1-a\overline{a}')}\left(-\gamma d^{(1)}_{2,0} + \frac{\beta}{2} d^{(1)}_{1,1} - \frac{\gamma}{2} d^{(2)}_{1,1} + \beta d^{(2)}_{0,2}\right). \tag{34}$$

The partial with respect to a_2 can be obtained in a similar fashion. In this case, the $(2, 1)$ entry and $(3, 2)$ entries are added as they arise in (31) and then (28). Again using (30), (32), (33) and simplifying using (29), we find that (35)

$$\frac{\partial}{\partial a_2}\log\det J = \frac{3\overline{a}_2}{1-a\overline{a}'} + \frac{2}{a\overline{a}'(1-a\overline{a}')}\left(\overline{\beta}d^{(1)}_{2,0} - \frac{\alpha}{2}d^{(1)}_{1,1} + \frac{\overline{\beta}}{2}d^{(2)}_{1,1} - \alpha d^{(2)}_{0,2}\right).$$

Since $\det J$ is holomorphic in a_1 and a_2, $\frac{\partial}{\partial\overline{a}_1}\log\det J = \frac{\partial}{\partial\overline{a}_2}\log\det J = 0$. Thus equations (34) and (35) give the total differential:

$$\begin{aligned} d\log\det J &= \frac{\partial}{\partial a_1}(\log\det J)da_1 + \frac{\partial}{\partial a_2}(\log\det J)da_2 \\ &= \frac{3(\overline{a}_1 da_1 + \overline{a}_2 da_2)}{1-a\overline{a}'} + \frac{2}{a\overline{a}'(1-a\overline{a}')} \\ &\quad\times\left[(-\gamma da_1 + \overline{\beta}da_2)\left(d^{(1)}_{2,0} + \frac{1}{2}d^{(2)}_{1,1}\right) + (\beta da_1 - \alpha da_2)\left(d^{(1)}_{0,2} + \frac{1}{2}d^{(1)}_{1,1}\right)\right]. \end{aligned}$$

By taking the straight line path from $(0,0)$ to the fixed point (z_1, z_2) in B^2 with the point on the line being given by $\rho z = \rho(z_1, z_2) = (\rho z_1, \rho z_2)$, $0 \le \rho \le 1$, i.e., $a_1 = \rho z_1$, $a_2 = \rho z_2 (z\overline{z}' = |z_1|^2 + |z_2|^2$, $\sqrt{z\overline{z}'} = |z|)$, we obtain

$$\frac{d}{d\rho}\log\det J = \frac{\partial}{\partial a_1}(\log\det J)\frac{da_1}{d\rho} + \frac{\partial}{\partial a_2}(\log\det J)\frac{da_2}{d\rho}.$$

Thus

$$\frac{d}{d\rho}(\log\det J)d\rho = \frac{\partial}{\partial a_1}(\log\det J)da_1 + \frac{\partial}{\partial a_2}(\log\det J)da_2.$$

Hence,

$$\begin{aligned} \log\det J &= \int_0^1 \frac{d}{d\rho}(\log\det J)\,d\rho \\ &= \int_{0,0}^{(z_1,z_2)} \frac{\partial}{\partial a_1}(\log\det J)\,da_1 + \frac{\partial}{\partial a_2}(\log\det J)da_2 \\ &= 3\int_{0,0}^{(z_1,z_2)} \frac{(\overline{a}_1 da_1 + \overline{a}_2 da_2)}{(1-a\overline{a}')} \\ &\quad + 2\int_{(0,0)}^{(z_1,z_2)} \frac{(d^{(1)}_{2,0} + \frac{1}{2}d^{(2)}_{1,1}(-\gamma da_1 + \overline{\beta}da_2) + (d^{(2)}_{0,2} + \frac{1}{2}d^{(1)}_{1,1})(\beta da_1 - \alpha da_2)}{a\overline{a}'(1-a\overline{a}')}. \end{aligned}$$

Using (15) we obtain

$$\begin{aligned} \log\det J &= 3\int_0^1 \frac{\rho(z\overline{z}')}{1-\rho^2 z\overline{z}'}\,d\rho \\ &\quad - 2\int_0^1 \rho^2 z\overline{z}'\frac{[z_1(d^{(1)}_{2,0} + \frac{1}{2}d^{(2)}_{1,1}) + z_2(d^{(2)}_{0,2} + \frac{1}{2}d^{(1)}_{1,1})]}{\rho^2 z\overline{z}'(1-\rho^2 z\overline{z}')}\,d\rho \\ &= -\frac{3}{2}\log(1-z\overline{z}') - 2\int_0^1 \frac{z_1(d^{(1)}_{2,0} + \frac{1}{2}d^{(2)}_{1,1}) + z_2(d^{(2)}_{0,2} + \frac{1}{2}d^{(1)}_{1,1})}{1-\rho^2 z\overline{z}'}\,d\rho. \end{aligned}$$

Thus, we have the desired expression
(36)

$$\log\det J = \log\frac{1}{(1-z\overline{z}')^{3/2}} - 2\int_0^1 \frac{z_1(d^{(1)}_{2,0} + \frac{1}{2}d^{(2)}_{1,1}) + z_2(d^{(2)}_{0,2} + \frac{1}{2}d^{(1)}_{1,1})}{1-\rho^2 z\overline{z}'}\,d\rho.$$

To determine an upper bound for the integrand on the right-hand side of (36) we make a rotation of B^2, i.e., a transformation of the domain by a unitary matrix U. Let $\xi = (\xi_1, \xi_2) \in B^2$ and

$$U = \begin{pmatrix} U_{11}, & U_{12} \\ U_{21}, & U_{22} \end{pmatrix}$$

be a unitary matrix. Consider $\zeta = \xi U$. The series of $f(\zeta)$ has the form

$$f(\zeta) = \zeta + (\zeta_1^2, \zeta_1\zeta_2, \zeta_2^2) \begin{pmatrix} d_{2,0}^{(1)}, & d_{2,0}^{(2)} \\ d_{1,1}^{(1)}, & d_{1,1}^{(2)} \\ d_{0,2}^{(1)}, & d_{0,2}^{(2)} \end{pmatrix} + \cdots .$$

We define g by $g(\xi) = f(\xi U)$. Then $g(\xi)\overline{U}'$ will be a new normalized convex mapping with coefficients of second-order terms being defined as follows:

$$g(\xi)\overline{U}' = \xi + (\xi_1^2, \xi_1\xi_2, \xi_2^2) \begin{pmatrix} D_{2,0}^{(1)}, & D_{2,0}^{(2)} \\ D_{1,1}^{(1)}, & D_{1,1}^{(2)} \\ D_{0,2}^{(1)}, & D_{0,2}^{(2)} \end{pmatrix} + \cdots .$$

Substitution of $\zeta = \xi U$ shows the coefficient matrices are related in the following way:

$$\begin{pmatrix} D_{2,0}^{(1)}, & D_{2,0}^{(2)} \\ D_{1,1}^{(1)}, & D_{1,1}^{(2)} \\ D_{0,2}^{(1)}, & D_{0,2}^{(2)} \end{pmatrix} = \begin{pmatrix} U_{11}^2, & U_{11}U_{12}, & U_{12}^2 \\ 2U_{11}U_{21}, & U_{11}U_{22} + U_{21}U_{12}, & 2U_{12}U_{22} \\ U_{21}^2, & U_{21}U_{22}, & U_{22}^2 \end{pmatrix} \times \begin{pmatrix} d_{2,0}^{(1)}, & d_{2,0}^{(2)} \\ d_{1,1}^{(1)}, & d_{1,1}^{(2)} \\ d_{0,2}^{(1)}, & d_{0,2}^{(2)} \end{pmatrix} \overline{U}'.$$

From this equation and the fact that $\overline{U}'U = I$, we obtain

$$D_{2,0}^{(1)} + \frac{1}{2}D_{1,1}^{(2)} = \left(d_{2,0}^{(1)} + \frac{1}{2}d_{1,1}^{(2)}\right) U_{11} + \left(d_{0,2}^{(2)} + \frac{1}{2}d_{1,1}^{(1)}\right) U_{12}. \tag{37}$$

Since

$$\begin{bmatrix} z_1/|z| & z_2/|z| \\ -\overline{z}_2/|z| & \overline{z}_1/|z| \end{bmatrix}$$

is a unitary matrix, we can take $U_{11} = z_1/|z|$ and $U_{12} = z_2/|z|$. The hypothesis of the theorem gives that the class S is M-invariant and that X comes from A by normalization of the mappings of S. From the definition of $C(X)$ given in (5), we then have

$$C(X) \geq \{|D_{2,0}^{(1)} + (1/2)D_{1,1}^{(2)}|\}. \tag{38}$$

Using (38) and (36) in (37), we find that

$$|\log \det J(1 - z\overline{z}')] \leq 2C(X) \int_0^1 \frac{|z|\, d\rho}{1 - \rho z\overline{z}'} = C(X) \log \frac{1 + |z|}{1 - |z|}. \tag{39}$$

From (39), the other conclusions of the theorem follow:

$$\frac{(1 - |z|)^{C(X) - 3/2}}{(1 + |z|)^{C(X) + 3/2}} \leq |\det J| \leq \frac{(1 + |z|)^{C(X) - 3/2}}{(1 - |z|)^{C(X) + 3/2}} \tag{40}$$

and

$$|\arg\det J| \le C(X)\log\left(\frac{1+|z|}{1-|z|}\right).$$

4. Estimates for the second-order coefficients for convex mappings

We now obtain an explicit estimate of $C(X)$ for a class Cartan suggested studying: the convex mappings. We use the characterization for convex mappings obtained by Kikuchi in [6]. He showed that f given by (2) is convex in B^2 if and only if

$$\operatorname{Re}\left\{1-\overline{z}\left(\frac{df}{dz}\right)^{-1}\frac{d^2f}{dz^2}(\alpha^2)'\right\}>0 \tag{41}$$

holds for every unit vector $\alpha=(\alpha_1, \alpha_2)$ which satisfies

$$\operatorname{Re}\{\overline{z}\alpha'\}=0,$$

where $df/dz=J$, $\alpha^2=(\alpha_1^2, \alpha_1\alpha_2, \alpha_2\alpha_1, \alpha_2^2)$ and

$$\frac{d^2f}{dz^2}=\begin{pmatrix}\dfrac{\partial^2 f_1}{\partial z_1^2} & \dfrac{\partial^2 f_1}{\partial z_1\partial z_2} & \dfrac{\partial^2 f_1}{\partial z_2\partial z_1} & \dfrac{\partial^2 f_1}{\partial z_2^2}\\ \dfrac{\partial^2 f_2}{\partial z_1^2} & \dfrac{\partial^2 f_2}{\partial z_1\partial z_2} & \dfrac{\partial^2 f_2}{\partial z_2\partial z_1} & \dfrac{\partial^2 f_2}{\partial z_2^2}\end{pmatrix}.$$

Using the expansion of f in (2), (41) is equivalent to

$$\operatorname{Re}\left\{1-\overline{z}\begin{pmatrix}2d_{2,0}^{(1)} & d_{1,1}^{(1)} & d_{1,1}^{(1)} & 2d_{0,2}^{(1)}\\ 2d_{2,0}^{(2)} & d_{1,1}^{(2)} & d_{1,1}^{(2)} & 2d_{0,2}^{(2)}\end{pmatrix}\begin{pmatrix}\alpha_1^2\\ \alpha_1\alpha_2\\ \alpha_2\alpha_1\\ \alpha_2^2\end{pmatrix}+\cdots\right\}>0. \tag{42}$$

If we let b_1 and b_2 be positive real numbers and $\alpha=(\alpha_1, \alpha_2)$ where

$$\alpha_1=\frac{ib_1z_1}{\sqrt{b_1^2|z_1|^2+b_2^2|z_2|^2}}, \quad \alpha_2=\frac{ib_2z_2}{\sqrt{b_1^2|z_1|^2+b_2^2|z_2|^2}},$$

then $\operatorname{Re}\{\overline{z}\alpha'\}=0$. We find by (42) that

$$\begin{aligned}\operatorname{Re}\Bigg\{1&+\left(\frac{1}{b_1^2|z_1|^2+b_2^2|z_2|^2}\right)\overline{z}\\ &\times\begin{pmatrix}2d_{2,0}^{(1)} & d_{1,1}^{(1)} & d_{1,1}^{(1)} & 2d_{0,2}^{(1)}\\ 2d_{2,0}^{(2)} & d_{1,1}^{(2)} & d_{1,1}^{(2)} & 2d_{0,2}^{(1)}\end{pmatrix}\begin{pmatrix}b_1^2z_1^2\\ b_1b_2z_1z_2\\ b_2b_1z_1z_2\\ b_2^2z_2^2\end{pmatrix}+\cdots\Bigg\}>0.\end{aligned} \tag{43}$$

We represent $z=(z_1, z_2)$ as $\xi\eta$ where ξ is a complex scalar of magnitude less than one and $\eta=(\eta_1, \eta_2)$ satisfies

$$|\eta_1|^2+|\eta_2|^2=1. \tag{44}$$

Then (43) becomes

$$\text{Re}\left\{1+\frac{\xi(\overline{\eta}_1,\overline{\eta}_2)}{b_1^2|\eta_1|^2+b_2^2|\eta_2|^2} \times \begin{pmatrix} 2d_{2,0}^{(1)} & d_{1,1}^{(1)} & d_{1,1}^{(1)} & 2d_{0,2}^{(1)} \\ 2d_{2,0}^{(2)} & d_{1,1}^{(2)} & d_{1,1}^{(2)} & 2d_{0,2}^{(2)} \end{pmatrix} \begin{pmatrix} b_1^2\eta_1^2 \\ b_1b_2\eta_1\eta_2 \\ b_2b_1\eta_1\eta_2 \\ b_2^2\eta_2^2 \end{pmatrix} + \cdots \right\} > 0. \tag{45}$$

The left-hand side of (45) is the real part of an analytic function of one complex variable ξ in the open unit disk. Consider the series expansion about the origin. The coefficient of ξ can be represented with a probability measure ν on $0 \le t \le 2\pi$ as follows:

$$\begin{aligned}\frac{(\overline{\eta}_1,\overline{\eta}_2)}{b_1^2|\eta_1|^2+b_2^2|\eta_2|^2}&\begin{pmatrix} 2b_1^2\eta_1^2d_{2,0}^{(1)}+2b_1b_2\eta_1\eta_2d_{2,1}^{(1)}+2b_2^2\eta_2^2d_{0,2}^{(1)} \\ 2b_1^2\eta_1^2d_{2,0}^{(2)}+2b_1b_2\eta_1\eta_2d_{1,1}^{(2)}+2b_2^2\eta_2^2d_{0,2}^{(2)} \end{pmatrix} \\ &= 2\int_0^{2\pi} e^{-it}\,d\nu(t).\end{aligned}$$

Thus we have

$$\begin{aligned} &b_1^2\eta_1|\eta_1|^2d_{2,0}^{(1)}+b_1b_2|\eta_1|^2\eta_2d_{1,1}^{(1)}+b_2^2\overline{\eta}_1\eta_2^2d_{0,2}^{(1)}+b_1^2\eta_1^2\overline{\eta}_2d_{2,0}^{(2)} \\ &\quad + b_1b_2\eta_1|\eta_2|^2d_{1,1}^{(2)}+b_2^2\eta_2|\eta_2|^2d_{0,2}^{(2)} = (b_1^2|\eta_1|^2+b_2^2|\eta_2|^2)\int_0^{2\pi} e^{-it}\,d\nu(t). \end{aligned} \tag{46}$$

We note here that using Suffridge's characterization [8] leads to this same equation. Let $\eta_1 = |\eta_1|e^{i\theta_1}$ and $\eta_2 = |\eta_2|e^{i\theta_2}$. Multiplying (46) by $\overline{\eta}_1$ and integrating with respect to $d\theta_1/2\pi$ we find that

$$|b_1^2|\eta_1|^4d_{2,0}^{(1)}+b_1b_2|\eta_1|^2|\eta_2|^2d_{1,1}^{(2)}| \le (b_1^2|\eta_1|^2+b_2^2|\eta_2|^2)|\eta_1|. \tag{47}$$

Multiplying (46) by $\overline{\eta}_2$ and integrating with respect to $d\theta_2/2\pi$ we find that

$$|b_1b_2|\eta_1|^2|\eta_2|^2d_{1,1}^{(1)}+b_2^2|\eta_2|^4d_{0,2}^{(2)}| \le (b_1^2|\eta_1|^2+b_2^2|\eta_2|^2)|\eta_2|. \tag{48}$$

From symmetry and the discussion after (37) we need only consider (47) which becomes

$$|d_{2,0}^{(1)}+(b_2/b_1)|\eta_2/\eta_1|^2d_{1,1}^{(2)}| \le \frac{1}{|\eta_1|}[1+(b_2/b_1)^2|\eta_2/\eta_1|^2]. \tag{49}$$

Let $(b_2/b_1)|\eta_2/\eta_1|^2 = c$ and $|\eta_1| = x_1$. A straightforward argument determines the minimum of $\frac{1}{x}+c^2\frac{x}{1-x^2}$ for $0 < x < 1$. This minimum gives an explicit, but involved, upper bound $T(c)$ for $|d_{2,0}^{(1)}+cd_{1,1}^{(2)}|$ in terms of c. For our purpose we only need $T(c)$ for $c = 1/2$. For

$$(b_2/b_1)|\eta_2/\eta_1|^2 = 1/2,$$

equation (44) shows that (49) becomes

$$|d_{2,0}^{(1)}+(1/2)d_{1,1}^{(2)}| \le \frac{1}{|\eta_1|}+\frac{1}{4}\frac{|\eta_1|}{1-|\eta_1|^2}.$$

Since the function

$$\frac{1}{x} + \frac{1}{4}\frac{x}{1-x^2}$$

is minimized at $x = \sqrt{(9-\sqrt{33})/6}$, we obtain the estimate

$$|d^{(1)}_{2,0} + (1/2)d^{(2)}_{1,1}| < 1.761. \tag{50}$$

In the notation of the theorem, if we let $X = K$, the class of normalized convex functions on B^2, then (50) shows that

$$C(K) < 1.761. \tag{51}$$

The theorem implies that for $z \neq 0$,

$$|\log[(\det J)(1 - z\overline{z}')^{3/2}]| < 1.761 \log\left(\frac{1+|z|}{1-|z|}\right), \tag{52}$$

$$\frac{(1-|z|)^{0.261}}{(1+|z|)^{3.261}} < |\det J| < \frac{(1+|z|)^{0.261}}{(1-|z|)^{3.261}}, \tag{53}$$

and

$$|\arg\det J| < 1.761 \log\left(\frac{1+|z|}{1-|z|}\right).$$

5. A conjecture and an example

Our work suggests the conjecture that if f is a normalized convex mapping on B^n, then

$$|\log[\det J(1 - z\overline{z}')^{(n+1)/2}]| \leq \frac{n+1}{2} \log\left(\frac{1+|z|}{1-|z|}\right). \tag{54}$$

Inequality (54) would have as consequences that

$$\frac{1}{(1+|z|)^{n+1}} \leq |\det J| \leq \frac{1}{(1-|z|)^{n+1}} \tag{55}$$

and

$$|\arg\det J| \leq \frac{n+1}{2} \log\left(\frac{1+|z|}{1-|z|}\right). \tag{56}$$

For $n = 1$, it is known that (54) is sharp and (55) is sharp but (56) is not sharp. Hence, for $n > 1$, we do not expect (56) to be sharp.

For $n = 2$, the conjecture (54) is of the same form and close to the proved result (52). The consequence (55) of the conjecture would be

$$(1+|z|)^{-3} \leq |\det J| \leq (1-|z|)^{-3} \tag{57}$$

which is of a simpler form but still close to the proved inequality (53).

Still considering $n = 2$, another implication of the conjectured inequality (54) would be an inequality for $C(K)$. If, as z tends to zero, we consider the first-order terms in the identity (36) in light of the discussion after (37), then the equation (54) would imply that $C(K) \leq 3/2$. The example to be discussed proves that $C(K) \geq 3/2$. Hence, if the conjecture holds, then

$$C(K) = 3/2. \tag{58}$$

Conversely, if (58) holds, the theorem implies the conjecture (54) for $n = 2$.

Finally we remark that for n a positive integer, the following example shows that if conjecture (54) is true, it is sharp. Similarly, the example shows that if (55) holds, then it is sharp. We first consider the case $n = 2$. Define $f = (f_1, f_2)$ by

$$(59) \qquad (f_1, f_2) = \left(\frac{z_1}{1-z_1}, \frac{z_2}{1-z_1}\right) = (z_1 + z_1^2 + \cdots, z_2 + z_1 z_2 + \cdots).$$

Then $d_{2,0}^{(1)} + (1/2)d_{1,1}^{(2)} = 3/2$ and the Jacobian J for this function is given by

$$(60) \qquad J = \begin{pmatrix} \frac{1}{(1-z_1)^2} & 0 \\ \frac{z_2}{(1-z_1)^2} & \frac{1}{(1-z_1)} \end{pmatrix}.$$

We will give two proofs that the mapping f is convex. We generated the mapping as a limit of renormalized Möbius transformations of the identity mapping. Since the full class of convex mappings is M-invariant and the subset of normalized convex mappings is compact, the limit function exists and is convex. Consider the identity mapping (z_1, z_2) on B^2 and the point $a = (a_1, 0)$ in B^2 with $0 < a_1 < 1$. If we let $\phi_a(z)$ be the Möbius transformation taking a into the origin, then from (7) ϕ_a is given by

$$\phi_a(z) = \frac{a - z}{1 - z\bar{a}'} A,$$

where in this case equation (8) shows that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-a_1^2} \end{pmatrix}.$$

Then $\phi_a(z)$ can be calculated from these equations or from (9) with $w = z$ and $a = (a_1, 0)$. We find that

$$\phi_a(z) = \left(\frac{a_1 - z_1}{1 - a_1 z_1}, \frac{-z_2\sqrt{1-a_1^2}}{1 - a_1 z_1}\right).$$

The Jacobian is easily found:

$$J_\phi(0) = \begin{pmatrix} a_1^2 - 1 & 0 \\ 0 & -\sqrt{1-a_1^2} \end{pmatrix}.$$

Thus the renormalized mapping is given by

$$[\phi_a(z) - \phi_a(0)]J_\phi'^{-1}(0) = \left(\frac{a_1 - z_1}{1 - z_1 a_1} - a_1, \frac{-z_2\sqrt{1-a_1^2}}{1 - z_1 a_1}\right)\begin{pmatrix} \frac{-1}{1-a_1^2} & 0 \\ 0 & \frac{-1}{\sqrt{1-a_1^2}} \end{pmatrix}$$

$$= \left(\frac{z_1}{1 - a_1 z_1}, \frac{z_2}{1 - a_1 z_1}\right)$$

which approaches (59) as a_1 approaches 1 as claimed. By construction it is clear that the resulting function is convex. In the general case $n \geq 2$ a similar construction gives the convex mapping $f = (f_1, f_2, \ldots, f_n)$ defined by

$$(f_1(z), f_2(z), \ldots, f_n(z)) = \left(\frac{z_1}{1-z_1}, \frac{z_2}{1-z_1}, \frac{z_3}{1-z_1}, \ldots, \frac{z_n}{1-z_1}\right).$$

The Jacobian of this example is triangular; thus the determinant of the Jacobian is easily calculated. The result is

$$\det J_f(z) = \frac{z_1}{(1-z_1)^{n+1}}.$$

Consequently, if the conjectured inequalities (57) are true, they are sharp.

Returning to the case $n = 2$, we give a second proof that f given by (59) is convex. We consider the Kikuchi criterion (41) with

$$\alpha = i(A_1 z_1, A_2 z_2)/[A_1^2|z_1|^2 + A_2^2|z_2|^2]^{1/2}$$

where A_1 and A_2 are possible. The left side of (41) becomes the real part of

$$\left\{ A_1^2|z_1|^2 + A_2^2|z_2|^2 + (\bar{z}_1, \bar{z}_2)\begin{pmatrix} (1-z_1)^2, & 0 \\ -z_2(1-z_1), & 1-z_1 \end{pmatrix} \right.$$

$$\left. \times \begin{pmatrix} \dfrac{2}{(1-z_1)^3}, & 0, & 0, & 0 \\ \dfrac{2z_2}{(1-z_1)^3}, & \dfrac{1}{(1-z_1)^2}, & \dfrac{1}{(1-z_1)^2}, & 0 \end{pmatrix} \begin{pmatrix} A_1^2 z_1^2 \\ A_1 A_2 z_1 z_2 \\ A_2 A_1 z_2 z_1 \\ A_2^2 z_2^2 \end{pmatrix} \right\}.$$

Hence,

$$\text{(61)} \qquad \operatorname{Re}\left\{ A_1^2|z_1|^2 + A_2^2|z_2|^2 + \frac{2A_1^2 z_1|z_1|^2 + 2A_1A_2 z_1|z_2|^2}{1-z_1} \right\} \geq 0.$$

The left side of (61) equals

$$\text{(62)} \quad \operatorname{Re}\left\{ A_1^2|z_1|^2 \left(\frac{1+z_1}{1-z_1}\right) + A_2^2|z_2|^2 - A_1A_2|z_2|^2 + A_1A_2|z_2|^2\left(\frac{1+z_1}{1-z_1}\right) \right\}.$$

Using the method of completing the square and the fact that for $|z_1| < 1$

$$\operatorname{Re}\left\{\frac{1+z_1}{1-z_1}\right\} \geq \frac{1-|z_1|}{1+|z_1|},$$

one can show that the nonnegativity of (62) follows from the nonnegativity of

$$\text{(63)} \qquad A_1^2|z_1|^2\left[\frac{1-(|z_1|^2+|z_2|^2)}{1+|z_1|}\right] + |z_2|(1+|z_1|)\left(A_2 - \frac{|z_1|}{1+|z_1|}A_1\right)^2.$$

The expression (63) is nonnegative for all z such that $|z_1|^2 + |z_2|^2 \leq 1$ and the result follows.

For this convex mapping, the inequality (41) for the Kikuchi criterion is sharp at essentially every point of the boundary of B^2. Consider $z = (z_1, z_2)$ on the boundary. The first term of (63) is zero. And, for any positive choice of A_1, the choice of $A_2 = A_1|z_1|/(1+|z_1|)$ makes the second term of (63) equal to zero. Except for the point $z = (1, 0)$, the steps can be reversed to see that equality holds in the Kikuchi criterion (41). Furthermore, the exceptional point $(1, 0)$ is carried to infinity by the mapping.

This example was essentially known. Consider the Cayley transform [7, p. 31] of B^n onto an "upper half plane" in $\mathbb{C}^n$. By composing with an appropriate unitary transformation of the domain and a complex affine trnasformation of the range, a Cayley transform can be changed into a normalized mapping which is exactly our example.

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